Problem Set 3

Problem 1. Suppose that $K \subset S^3$, $n$ is a positive integer. Prove that the set
$$\{s \in \text{Spin}^c(S^3_K(n)) \mid HF_{\text{red}}(S^3_K(n), s) \neq 0\}$$
has at most $2g(K)-1$ elements. In particular, if $Y$ is a rational homology sphere, and there are exactly $N$ Spin$^c$ structures $s \in \text{Spin}^c(Y)$ satisfying $HF_{\text{red}}(Y, s) \neq 0$, then $Y$ cannot be obtained by integer surgery on any knot in $S^3$ with genus $\leq \frac{N}{2}$.

Problem 2. Let $K \subset S^3$ be an L-space knot, $C = CFK^\infty(S^3, K)$, $k \in \mathbb{Z}$.
(1) Prove that $H_*(C\{i < 0, j \geq k\}) \cong \mathbb{Z}[U^{-1}, \ldots, U^{1-t}]$ for some integer $t \geq 0$.
(2) Prove
$$\chi(C\{i < 0, j \geq k\}) = t_k = \sum_{n=1}^{\infty} n \alpha_{n+k},$$
where $\alpha_i$’s are the coefficients of the normalized Alexander polynomial.
(3) Prove $t = t_k$.

Problem 3. Let $K \subset S^3$ be an L-space knot, $C = CFK^\infty(S^3, K)$, $k \in \mathbb{Z}$.
(1) Prove that $H_*(C\{\max(i, j-k) = 0\}) \cong \mathbb{Z}$.
(2) Prove that $H_*(C\{i < 0, j = k\})$ is either 0 or $\mathbb{Z}$, the same is true for $H_*(C\{i = 0, j \leq k\})$.
(3) Prove that exactly one of the two groups $H_*(C\{i < 0, j = k\})$ and $H_*(C\{i = 0, j \leq k\})$ is $\mathbb{Z}$.
(4) Prove that if $H_*(C\{i = 0, j = k\}) \cong \mathbb{Z}^2$, then both $H_*(C\{i < 0, j = k\})$ and $H_*(C\{i \leq 0, j = k\})$ are $\mathbb{Z}$.
(5) Prove that $H_*(C\{i = 0, j = k\})$ is either 0 or $\mathbb{Z}$. As a consequence, the coefficients of the Alexander polynomial of an L-space knot are 0 or $\pm 1$. 

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Exercises for Lecture 3

1. Let \( V \cong \mathbb{R}^2 \) and \( \phi_j : V \rightarrow \mathbb{R} \) a non-zero homomorphism for \( 1 \leq j \leq n \). Define
   \[
   \| \cdot \| : V \rightarrow [0, \infty) \quad v \mapsto \sum_j |\phi_j(v)|
   \]
   (a) Show that \( \| \cdot \| \) is a (non-zero) seminorm.
   (b) If \( \| \cdot \| \) is a norm show that its ball of radius 1 is a finite-sided polygon each of whose vertices lies on one of the lines \( \ker(\phi_j) \).
   (c) Determine the ball of radius 1 of \( \| \cdot \| \) when it is not a norm.

2. If \( M \) is the trefoil exterior then \( \pi_1(M) = \langle \gamma_1, \gamma_2 : \gamma_1^2 = \gamma_2^3 \rangle \). The fundamental group of \( \partial M \) is generated by a meridian
   \[
   \mu = \gamma_1 \gamma_2^{-1}
   \]
   and the Seifert fibre class
   \[
   h = \gamma_1^2,
   \]
   which is central in \( \pi_1(M) \). We saw in Lecture 2 that \( X^{irr}(M) \) is a curve \( X_0 \) which has image \( \{(0,1,w) : w \in \mathbb{C} \} \cong \mathbb{C} \) under the map \( X_0 \rightarrow \mathbb{C}^3, \chi \mapsto (\chi(\gamma_1), \chi(\gamma_2), \chi(\gamma_1 \gamma_2)) \).
   (a) Show that for \( z \in \mathbb{C} \) there is a homomorphism \( \rho_z : \pi_1(M) \rightarrow SL(2,\mathbb{C}) \) given by
   \[
   \rho_z(\gamma_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \rho_z(\gamma_2) = \begin{pmatrix} z & -(z^2 - z + 1) \\ 1 & 1 - z \end{pmatrix}
   \]
   and that the map \( z \mapsto \chi_{\rho_z} \) parametrises \( X_0 \).
   (b) Show that \( \| h \|_{X_0} = 0 \) and \( \| \mu \|_{X_0} = 2^1 \) and deduce that if \( \alpha \in H_1(\partial M) \) then
   \[
   \| \alpha \|_{X_0} = 2|\alpha \cdot h|
   \]
   where \( \alpha \cdot h \) is the algebraic intersection of \( \alpha \) and \( h \) on \( \partial M \).

\(^1\text{Hint:} \) Since \( X_0 \cong \mathbb{C} \) it has a unique ideal point. Recall that \( \mu = \gamma_1 \gamma_2^{-1} \). Calculate the multiplicity of the ideal point as a pole of \( f_\mu : X_0 \rightarrow \mathbb{C} \) using the parametrisation of \( X_0 \) given above.
0.1 References for Lecture 3
